# Robustness of Sequential Probability Ratio Tests in Case of Nuisance Parameters

Karl-Heinz Eger[1] and Evgeni Borisovich Tsoy[2]
[1]Chemnitz University of Technology, Department of Mathematics, Chemnitz, Germany
[2]Novosibirsk State Technical University, Department of Applied Mathematics, Novosibirsk, Russia

***Abstract*—This paper deals with the computation of OC- and ASN-function of sequential probability ratio tests in the multi-parameter case. In generalization of the method of conjugated parameter pairs Wald-like approximations are presented for the OC- and ASN-function. These characteristics can be used describing robustness properties of a sequential test in case of nuisance parameters. As examples tests are considered for the mean and the variance of a normal distribution.**

## I. Introduction

This paper deals with sequential probability ratio tests in the multi-parameter case in the following sense. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with density function $f_{\vec{\theta}}(x)$ with respect to some measure $\mu$. Let $\vec{\theta} = (\theta_1, \theta_2)$ be a two-dimensional parameter with values in a parameter space $\Theta$. Our aim is to discriminate between two simple one-dimensional hypotheses, e.g. for the parameter $\theta_1$,

$$H_0 : \theta_1 = \theta_{10} \quad \text{against} \quad H_1 : \theta_1 = \theta_{11}, \tag{1}$$

$\theta_{10} \neq \theta_{11}$, by means of a sequential probability ratio test based on the sequence $X_1, X_2, \ldots$.

We suppose that the second parameter $\theta_2$ is known and possesses the value $\theta_2 = \theta_{20}$. In this context the parameter $\theta_2$ is a so-called nuisance or disturbing parameter. Special variants of such tests are e.g. tests for the mean of a normal distribution with known variance, for instance in case of testing the mean with known measuring accuracy, or tests for the variance of a normal distribution with known mean, respectively. Other examples in this sense are one-dimensional tests for parameters of Weibull or gamma distributions.

Then the question arises, what happens with the statistical properties of our test if the true value of the nuisance parameter $\theta_2$ is different from the assumed value $\theta_{20}$ or how robust is our test in view of modifications of true value of nuisance parameter.

This paper presents a method how we can describe the robustness of a sequential probability ratio test in case of nuisance parameters by means of generalized Wald approximations for the operating characteristic function (OC-function) and average sample number function (ASN-function). A helpful tool in this context is the so-called principle of conjugated parameter pairs. Especially we will consider examples for testing the mean or the variance of a normal distribution in case of nuisance parameters.

## II. The sequential probability ratio test

We consider a sequential probability ratio test (SPRT) for discriminating between two simple two-dimensional hypotheses

$$H_0 : \theta_1 = \theta_{10}, \theta_2 = \theta_{20} \text{ and } H_1 : \theta_1 = \theta_{11}, \theta_2 = \theta_{21} \tag{2}$$

with $\theta_{10} \neq \theta_{11}$ or/and $\theta_{21} \neq \theta_{20}$. This test is a generalization of the test for hypotheses (1) considered above. For instance, we get the hypotheses (1) if we put $\theta_{20} = \theta_{21}$.

Let $L_{n,\vec{\theta}_0,\vec{\theta}_1}$ be the likelihood ratio on stage $n$, $n = 1, 2, \ldots$ then we have

$$L_{n,\vec{\theta}_0,\vec{\theta}_1} = \prod_{i=1}^{n} \frac{f_{\vec{\theta}_1}(X_i)}{f_{\vec{\theta}_0}(X_i)}.$$

To given stopping bounds $B$ and $A$, $0 < B < 1 < A < \infty$, the sample size $N$ and the terminal decision rule $\delta$ of Wald's sequential probability ratio test are defined by

$$N = \min\{n \geq 1 : L_{n,\vec{\theta}_0,\vec{\theta}_1} \notin (B, A)\}$$

and

$$\delta = 1_{\left\{L_{N,\vec{\theta}_0,\vec{\theta}_1} \leq B\right\}}.$$

That means, we continue observations for $n = 1, 2, \ldots$ as long as the critical inequality $B < L_{n,\vec{\theta}_0,\vec{\theta}_1} < A$ holds. If on observation stage $n$ for the first time $L_{n,\vec{\theta}_0,\vec{\theta}_1} \notin (B, A)$ and if then $L_{n,\vec{\theta}_0,\vec{\theta}_1} \leq B$ or $L_{n,\vec{\theta}_0,\vec{\theta}_1} \geq A$ holds we accept the hypothesis $H_0$ or $H_1$, respectively. We denote this SPRT by $S(B, A)$.

The most important characteristics with respect to a description of the statistical properties of our test are the operating characteristic function (OC-function) $Q(\vec{\theta}) = E_{\vec{\theta}}\delta$, $\vec{\theta} \in \Theta$, and the average sample number function (ASN-function) $E_{\vec{\theta}}N$, $\vec{\theta} \in \Theta$.

If $P_{\vec{\theta}}(L_{1,\vec{\theta}_0,\vec{\theta}_1} = 1) < 1$ then we have $P_{\vec{\theta}}(N < \infty) = 1$ and $E_{\vec{\theta}}N < \infty$. Moreover, the Wald-Wolfowitz-Theorem holds. That means, the test $S(B, A)$ minimizes the average sample number function for $\vec{\theta} = \vec{\theta}_0$ and $\vec{\theta} = \vec{\theta}_1$ among all tests whose error probabilities are not greater than the error probabilities of Wald's SPRT at $\vec{\theta} = \vec{\theta}_0$ and $\vec{\theta} = \vec{\theta}_1$.

The general problem of Wald's SPRT consists in the computation of its characteristics, e.g., the OC-function or the ASN-function. We will demonstrate that the so-called method of conjugate parameter pairs (see [3]) can be extended to the case considered here obtaining Waldlike approximations for the OC- and ASN-function.

### A. The Wald approximations

The OC- and ASN-function of test $S(B, A)$ can be computed approximately in sense of the so-called Wald approximations by means of conjugated parameter pairs as follows [3].

DEFINITION. Two parameter pairs $(\vec{\theta},\vec{\theta}')$ and $(\vec{\theta}_0,\vec{\theta}_1) \in \Theta\times\Theta$ are said to be *conjugated*, if a real number $h$, $h \neq 0$, exists, such that

$$L_{n,\vec{\theta},\vec{\theta}'} = L^h_{n,\vec{\theta}_0,\vec{\theta}_1}, \quad n = 1,2,...,$$

holds. We write: $(\vec{\theta},\vec{\theta}') \overset{h}{\sim} (\vec{\theta}_0,\vec{\theta}_1)$.

If $(\vec{\theta},\vec{\theta}') \overset{h}{\sim} (\vec{\theta}_0,\vec{\theta}_1)$ the OC-function $Q(\vec{\theta})$ and the power function $M(\vec{\theta}) = E_{\vec{\theta}}(1-\delta), \vec{\theta} \in \Theta$, of test $S(B,A)$ satisfy the relations

$$\frac{Q(\vec{\theta}')}{Q(\vec{\theta})} = E_{\vec{\theta}}\left(L^h_{N,\vec{\theta}_0,\vec{\theta}_1}|H_0 \text{ is acc.}\right) \leq B^h \tag{3}$$

and

$$\frac{M(\vec{\theta}')}{M(\vec{\theta})} = E_{\vec{\theta}}\left(L^h_{N,\vec{\theta}_0,\vec{\theta}_1}|H_1 \text{ is acc.}\right) \geq A^h, \tag{4}$$

where in case of

$$\begin{aligned} &P_{\vec{\theta}}(L_{N,\vec{\theta}_0,\vec{\theta}_1} = B|H_0 \text{ is accepted}) \\ &= P_{\vec{\theta}}(L_{N,\vec{\theta}_0,\vec{\theta}_1} = A|H_1 \text{ is accepted}) = 1 \end{aligned} \tag{5}$$

the equals signs hold. We remark, that in case of $P_{\vec{\theta}}(N < \infty) = 1$ (closed test) moreover $M(\vec{\theta}) = 1 - Q(\vec{\theta})$ holds. A sufficient condition for closeness is, for instance, $P_{\vec{\theta}}(L_{1,\vec{\theta}_0,\vec{\theta}_1} = 1) < 1$ (see e.g. [3]).

### *B. The OC-function*

For a closed test $S(B,A)$ we get in case of (5) by $(\vec{\theta},\vec{\theta}') \overset{h}{\sim} (\vec{\theta}_0,\vec{\theta}_1)$, (3) and (4) for the OC-function

$$Q(\vec{\theta}) = Q^*(\vec{\theta}) = \frac{A^h - 1}{A^h - B^h}$$

and

$$Q(\vec{\theta}') = Q^*(\vec{\theta}') = B^h Q^*(\vec{\theta}).$$

If condition (5) holds approximately, that means the excess over the stopping boundaries is negligible when the test ends, e.g., $L_{n,\vec{\theta}_0,\vec{\theta}_1} \approx B$ or $L_{n,\vec{\theta}_0,\vec{\theta}_1} \approx A$ when $N = n$ then we have $Q(\vec{\theta}) \approx Q^*(\vec{\theta})$ and $Q(\vec{\theta}') \approx Q^*(\vec{\theta}') = B^h Q^*(\vec{\theta})$. This are the famous WALD approximations for the OC-function. If to given $\vec{\theta}$ no $h \neq 0$ and no parameter vector $\vec{\theta}' \neq \vec{\theta}$ exist such that $(\vec{\theta},\vec{\theta}') \overset{h}{\sim} (\vec{\theta}_0,\vec{\theta}_1)$, e.g., in case of $E_{\vec{\theta}}Z_{1,\vec{\theta}_0,\vec{\theta}_1} = E_{\vec{\theta}} \ln L_{1,\vec{\theta}_0,\vec{\theta}_1} = 0$, the WALD approximation for the OC-function can be extended by $Q(\vec{\theta}) \approx Q^*(\vec{\theta}) = \ln A/(\ln A - \ln B)$ for $E_{\vec{\theta}}Z_{1,\vec{\theta}_0,\vec{\theta}_1} = 0$.

### *C. The stopping bounds*

Under condition (5) we obtain a test $S(B,A)$ at size $(\alpha,\beta)$, $0 < \alpha,\beta < 1$, $\alpha+\beta < 1$, that means $Q(\vec{\theta}_0) = 1-\alpha$ and $Q(\vec{\theta}_1) = \beta$, if the stopping boundaries $B$ and $A$ satisfy the conditions

$$B = B^* = \frac{1-\beta}{\alpha} \quad \text{and} \quad A = A^* = \frac{\beta}{1-\alpha}. \tag{6}$$

The values $B^*$ and $A^*$ are the so-called WALD approximations for the stopping boundaries.

A sufficient condition for an admissible test for the hypotheses (2) at size $(\alpha,\beta)$ is $B = \beta$ and $A = 1/\alpha$. Then we have $Q(\vec{\theta}_0) \geq 1-\alpha$ and $Q(\vec{\theta}_1) \leq \beta$.

### *D. The ASN-function*

By means of the moment equation $E_{\vec{\theta}}Z_{N,\vec{\theta}_0,\vec{\theta}_1} = E_{\vec{\theta}}N \cdot E_{\vec{\theta}}Z_{1,\vec{\theta}_0,\vec{\theta}_1}$ which holds for our tests if, e.g., $P_{\vec{\theta}}(L_{1,\vec{\theta}_0,\vec{\theta}_1} = 1) < 1$ we get in case of $E_{\vec{\theta}}Z_{1,\vec{\theta}_0,\vec{\theta}_1} \neq 0$ for the average sample number

$$\begin{aligned} &E_{\vec{\theta}}N = (E_{\vec{\theta}}(Z_{N,\vec{\theta}_0,\vec{\theta}_1}|H_0 \text{ is acc.})Q(\vec{\theta}) \\ &+E_{\vec{\theta}}(Z_{N,\vec{\theta}_0,\vec{\theta}_1}|H_1 \text{ is acc.})(1-Q(\vec{\theta}))/E_{\vec{\theta}}Z_{1,\vec{\theta}_0,\vec{\theta}_1}. \end{aligned}$$

If we again assume that condition (5) holds approximately we obtain the so-called WALD approximation $E^*_{\vec{\theta}}N$ for the average sample number $E_{\vec{\theta}}N$:

$$E_{\vec{\theta}}N \approx E^*_{\vec{\theta}}N = \frac{\ln BQ^*(\vec{\theta}) + \ln A(1-Q^*(\vec{\theta}))}{E_{\vec{\theta}}Z_{1,\vec{\theta}_0,\vec{\theta}_1}}. \tag{7}$$

In case of $E_{\vec{\theta}}Z_{1,\vec{\theta}_0,\vec{\theta}_1} = 0$ we get by means of the moment equation $E_{\vec{\theta}}Z^2_{N,\vec{\theta}_0,\vec{\theta}_1} = E_{\vec{\theta}}N \cdot E_{\vec{\theta}}Z^2_{1,\vec{\theta}_0,\vec{\theta}_1}$ the approximation $E_{\vec{\theta}}N \approx E^*_{\vec{\theta}}N = -\ln B \ln A/E_{\vec{\theta}}Z^2_{1,\vec{\theta}_0,\vec{\theta}_1}$.

### *E. Conjugated parameter pairs*

According to our definition of conjugated parameter pairs we have in the i.i.d. case the following criterion. It holds $(\vec{\theta},\vec{\theta}') \overset{h}{\sim} (\vec{\theta}_0,\vec{\theta}_1)$ if to a given parameter vector $\vec{\theta} \in \Theta$ a real number $h \neq 0$ and a parameter vector $\vec{\theta}' \in \Theta$, $\vec{\theta}' \neq \vec{\theta}$ exist such that

$$\frac{f_{\vec{\theta}'}(x)}{f_{\vec{\theta}}(x)} = \left(\frac{f_{\vec{\theta}_1}(x)}{f_{\vec{\theta}_0}(x)}\right)^h \tag{8}$$

holds for $x \in \mathcal{X}$, $\mathcal{X}$ the domain of $X_1$ . Hence, a necessary existence condition for conjugated parameter pairs is, that function

$$f_{\vec{\theta}'}(x) = \left(\frac{f_{\vec{\theta}_1}(x)}{f_{\vec{\theta}_0}(x)}\right)^h f_{\vec{\theta}}(x), \quad x \in \mathcal{X},$$

is a density function. Because of $f_{\vec{\theta}'}(x) \geq 0$ for $x \in \mathcal{X}$ we can compute a value $h$, $-\infty < h < \infty$, such that

$$\begin{aligned} \varphi_{\vec{\theta}}(h) = \int_{-\infty}^{\infty} f_{\vec{\theta}'}(x)dx = \int_{-\infty}^{\infty} \left(\frac{f_{\vec{\theta}_1}(x)}{f_{\vec{\theta}_0}(x)}\right)^h f_{\vec{\theta}}(x)\, dx \\ = E_{\vec{\theta}}\, \mathrm{e}^{hZ_{1,\vec{\theta}_0,\vec{\theta}_1}} = 1. \end{aligned}$$

The function $\varphi_{\vec{\theta}}(h)$ is as function of $h$, $-\infty < h < \infty$, the moment-generating function of the random variable $Z_{1,\vec{\theta}_0,\vec{\theta}_1} = \ln L_{1,\vec{\theta}_0,\vec{\theta}_1}$. It holds $\varphi_{\vec{\theta}}(0) = 1$, $\lim_{h\to\pm\infty} \varphi_{\vec{\theta}}(h) = \infty$, $\varphi'_{\vec{\theta}}(0) = E_{\vec{\theta}}Z_{1,\vec{\theta}_0,\vec{\theta}_1}$ as well as $\varphi''_{\vec{\theta}}(h) = E_{\vec{\theta}}(Z^2_{1,\vec{\theta}_0,\vec{\theta}_1}\, \mathrm{e}^{hZ_{1,\vec{\theta}_0,\vec{\theta}_1}}) > 0$. This means, that $\varphi_{\vec{\theta}}(h)$ is a convex function in $h$. Hence, we have in case of $E_{\vec{\theta}}Z_{1,\vec{\theta}_0,\vec{\theta}_1} < 0$ and $E_{\vec{\theta}}Z_{1,\vec{\theta}_0,\vec{\theta}_1} > 0$ beside the trivial solution $h = 0$ of equation $\varphi_{\vec{\theta}}(h) = 1$ always an unique solution $h > 0$ and $h < 0$, respectively.

If condition (5) holds approximately we have in case of $(\vec{\theta},\vec{\theta}') \overset{h}{\sim} (\vec{\theta}_0,\vec{\theta}_1)$

$$Q(\vec{\theta}) \approx Q^*_{\vec{\theta}}(\vec{\theta}) = \frac{A^h - 1}{A^h - B^h}.$$

An explicit determination of parameter vector $\vec{\theta}'$ is not necessary then.

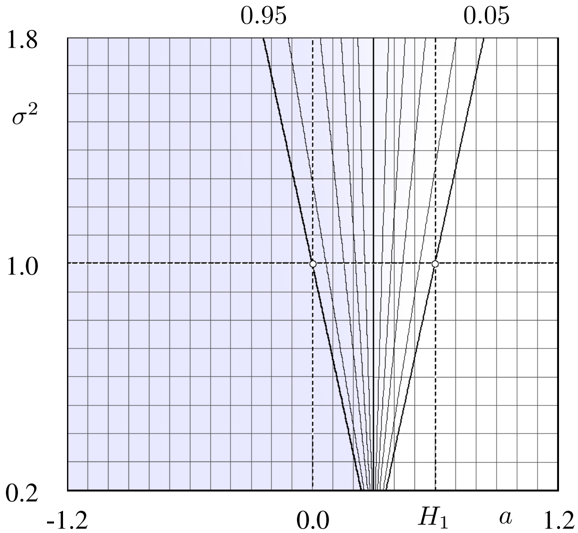


Fig. 1. Contour lines of OC-function $Q^*(a, \sigma^2) = q$ for $H_0 : a_0 = 0, \sigma_0^2 = 1, H_1 : a_1 = 0.6, \sigma_1^2 = 1,\ q = 0.05(0.05)0.95$.

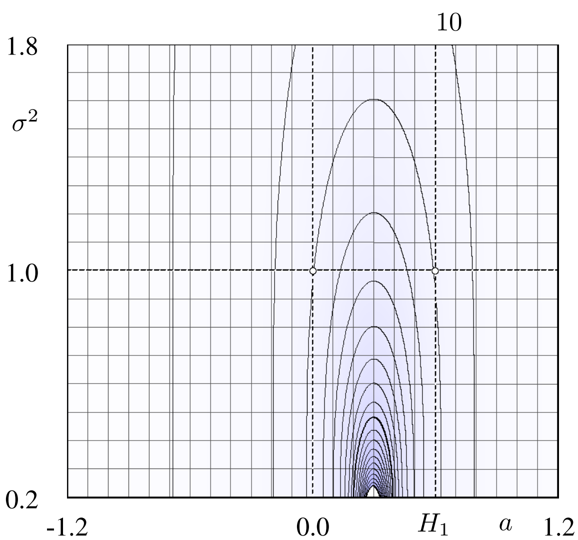


Fig. 2. Contour lines of ASN-function $E^*_{a,\sigma^2} N = e$ for $H_0 : a_0 = 0, \sigma_0^2 = 1, H_1 : a_1 = 0.6, \sigma_1^2 = 1,\ e = 0(5)100$.

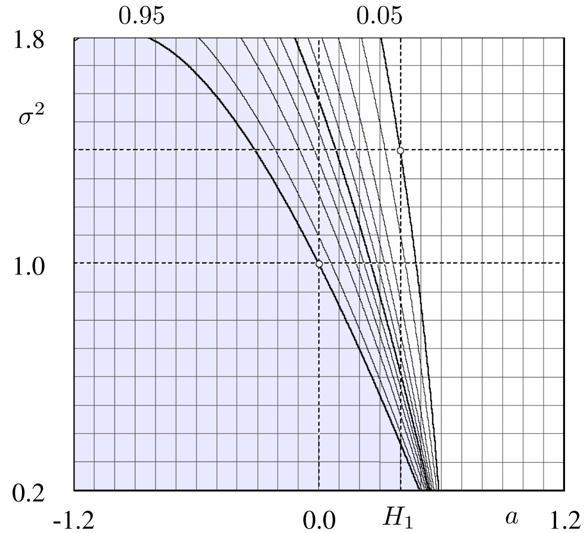


Fig. 3. Contour lines of OC-function $Q^*(a, \sigma^2) = q$ for $H_0 : a_0 = 0, \sigma_0^2 = 1, H_1 : a_1 = 0.4, \sigma_1^2 = 1.4,\ q = 0.05(0.05)0.95$.

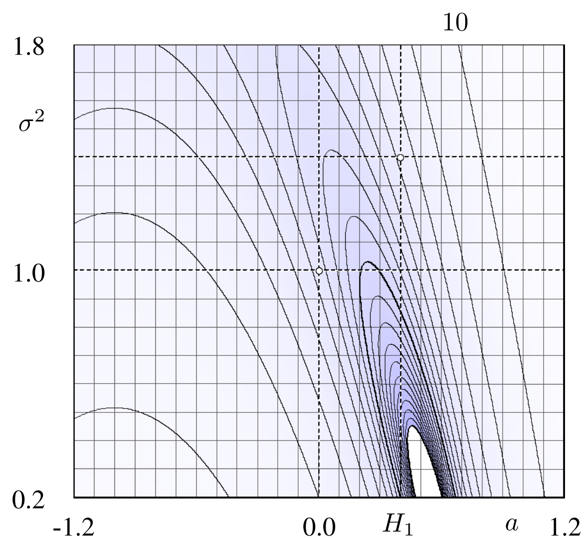


Fig. 4. Contour lines of ASN-function $E^*_{a,\sigma^2} N = e$ for $H_0 : a_0 = 0, \sigma_0^2 = 1, H_1 : a_1 = 0.4, \sigma_1^2 = 1.4,\ e = 0(5)100$.

## III. Example: Normal distribution

Let $X_1, X_2, \ldots$ be independent $N(a, \sigma^2)$-distributed random variables. We consider an SPRT for discriminating between the simple hypotheses

$$H_0 : a = a_0, \sigma^2 = \sigma_0^2 \quad \text{and} \quad H_1 : a = a_1, \sigma^2 = \sigma_1^2$$

where $a_0 \neq a_1$ and/or $\sigma_0^2 \neq \sigma_1^2$ holds. In case of $\sigma_0^2 = \sigma_1^2$, $\sigma_0^2$ known, we have the usual one-dimensional test for the mean with hypotheses

$$H_0 : a = a_0 \quad \text{and} \quad H_1 : a = a_1.$$

Analogously, we get in case of $a_0 = a_1$, $a_0$ known, the corresponding one-dimensional test for the variance with hypotheses

$$H_0 : \sigma^2 = \sigma_0^2 \quad \text{and} \quad H_1 : \sigma^2 = \sigma_1^2.$$

Let $f_{\vec{\theta}}(x) = f_{a,\sigma^2}(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-a)^2}{2\sigma^2}\right)$ be the density function of our observation variables $X_1, X_2, \ldots$. Then for the likelihood ratio for hypotheses (1) we obtain on stage $n$, $n = 1, 2, \ldots$,

$$\begin{aligned} L_{n,\vec{\theta}_0,\vec{\theta}_1} &= L_{n,a_0,\sigma_0^2,a_1,\sigma_1^2} = \prod_{i=1}^{n} \frac{f_{a_1,\sigma_1^2}(X_i)}{f_{a_0,\sigma_0^2}(X_i)} \\ &= \prod_{i=1}^{n} \exp\left(c_0 + c_1 X_i + c_2 X_i^2\right), \end{aligned}$$

where

$$c_0 = \ln \frac{\sigma_0}{\sigma_1} + \frac{1}{2}\left(\frac{a_0^2}{\sigma_0^2} - \frac{a_1^2}{\sigma_1^2}\right), \quad c_1 = \frac{a_1}{\sigma_1^2} - \frac{a_0}{\sigma_0^2} \tag{9}$$

and

$$c_3 = \frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right) \tag{10}$$

holds.

The Wald approximations for the OC- and ASN-function: Following the concept of conjugated parameter

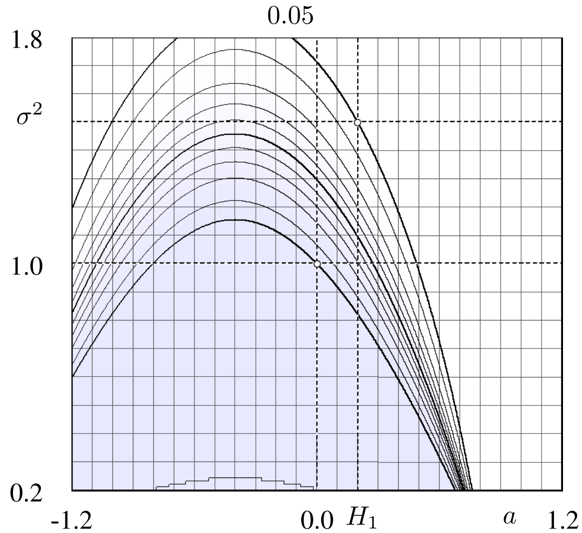


Fig. 5. Contour lines of OC-function $Q^*(a, \sigma^2) = q$ for $H_0 : a_0 = 0, \sigma_0^2 = 1, H_1 : a_1 = 0.2, \sigma_1^2 = 1.5,\ q = 0.05(0.05)0.95$.

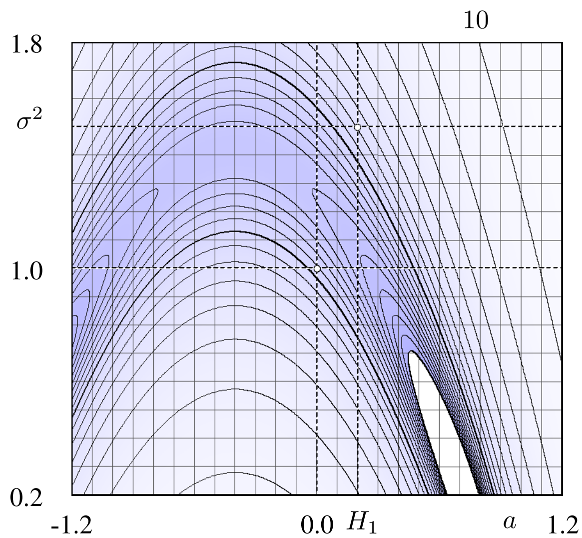


Fig. 6. Contour lines of ASN-function $E^*_{a,\sigma^2} N = e$ for $H_0 : a_0 = 0, \sigma_0^2 = 1, H_1 : a_1 = 0.2, \sigma_1^2 = 1.5,\ e = 0(5)100$.

pairs we have to calculate to given parameter values $a$ and $\sigma^2$ conjugated parameter values $a'$ and ${\sigma'}^2$ such that

$$L_{n,a,\sigma^2,a',\sigma'^2} = L^h_{n,a_0,\sigma_0^2,a_1,\sigma_1^2}, \quad n = 1, 2, ...$$

holds. Because of $X_1, X_2, ...$ are assumed to be i.i.d. random variables this condition is equivalent to

$$L_{1,a,\sigma^2,a',\sigma'^2} = L^h_{1,a_0\sigma_0^2,a_1,\sigma_1^2}. \tag{11}$$

Let $Z_{1,a,\sigma^2,a',\sigma'^2}$ and $Z_{1,a_0,\sigma_0^2,a_1,\sigma_1^2}$ be the logarithms of $L_{1,a,\sigma^2,a',\sigma'^2}$ and $L_{1,a_0\sigma_0^2,a_1,\sigma_1^2}$ then we have

$$Z_{1,a_0,\sigma_0^2,a_1,\sigma_1^2} = c_0 + c_1 X_1 + c_2 X_1^2$$

and

$$Z_{1,a,\sigma^2,a',\sigma'^2} = c'_0 + c'_1 X_1 + c'_2 X_1^2,$$

where $c'_0, c'_1$ and $c'_2$ are coefficients defined in accordance to (9) and (10) for the parameter set $a$, $\sigma^2$, $a'$ and ${\sigma'}^2$. Then (11) is further equivalent to

$$Z_{1,a,\sigma^2,a',\sigma'^2} = h Z_{1,a_0,\sigma_0^2,a_1,\sigma_1^2}$$

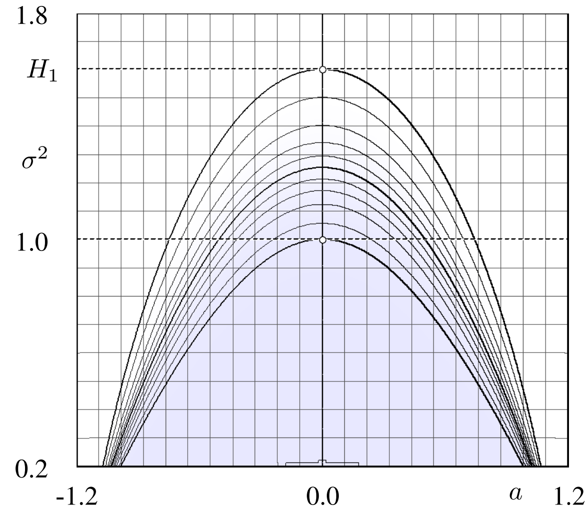


Fig. 7. Contour lines of OC-function $Q^*(a, \sigma^2) = q$ for $H_0 : a_0 = 0, \sigma_0^2 = 1, H_1 : a_1 = 0, \sigma_1^2 = 1.6,\ q = 0.05(0.05)0.95$.

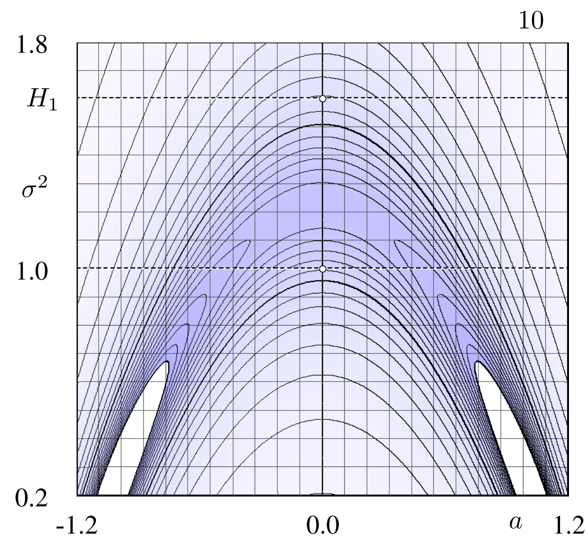


Fig. 8. Contour lines of ASN-function $E^*_{a,\sigma^2} N = e$ for $H_0 : a_0 = 0, \sigma_0^2 = 1, H_1 : a_1 = 0, \sigma_1^2 = 1.6,\ e = 0(5)100$.

or

$$c'_0 + c'_1 X_1 + c'_2 X_1^2 = h c_0 + h c_1 X_1 + h c_2 X_1^2.$$

By means of comparison of coefficients this implies $(a, \sigma^2, a', {\sigma'}^2) \overset{h}{\sim} (a_0, \sigma_0^2, a_1, \sigma_1^2)$ if

$$c'_0 = h c_0, \quad c'_1 = h c_1 \quad \text{und} \quad c'_0 = h c_2$$

holds. To given $a$ and $\sigma^2$ these three equations form a non-linear system of equations with unknowns $h$, $a'$ and ${\sigma'}^2$, which can be solved by means of an appropriate iteration procedure. With respect to the WALD approximations of OC- and ASN-function, we are looking for, it is sufficient to compute then only the parameter value $h$.

Alternatively, the parameter value $h$ could be computed as follows. It holds $(a, \sigma^2, a', {\sigma'}^2) \overset{h}{\sim} (a_0, \sigma_0^2, a_1, \sigma_1^2)$ if

$$f_{a',\sigma'^2}(x) = \left( \frac{f_{a_1,\sigma_1^2}(x)}{f_{a_0,\sigma_0^2}(x)} \right)^h f_{a,\sigma^2}(x)$$

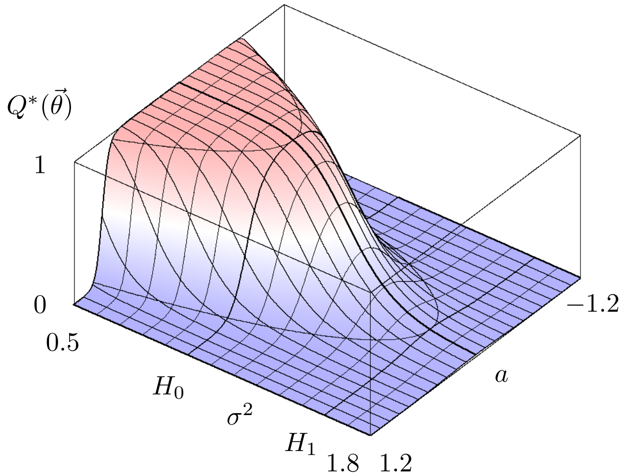


Fig. 9. OC-function $Q^*(a, \sigma^2)$ for $H_0 : a_0 = 0, \sigma_0^2 = 1, H_1 : a_1 = 0, \sigma_1^2 = 1.6$.

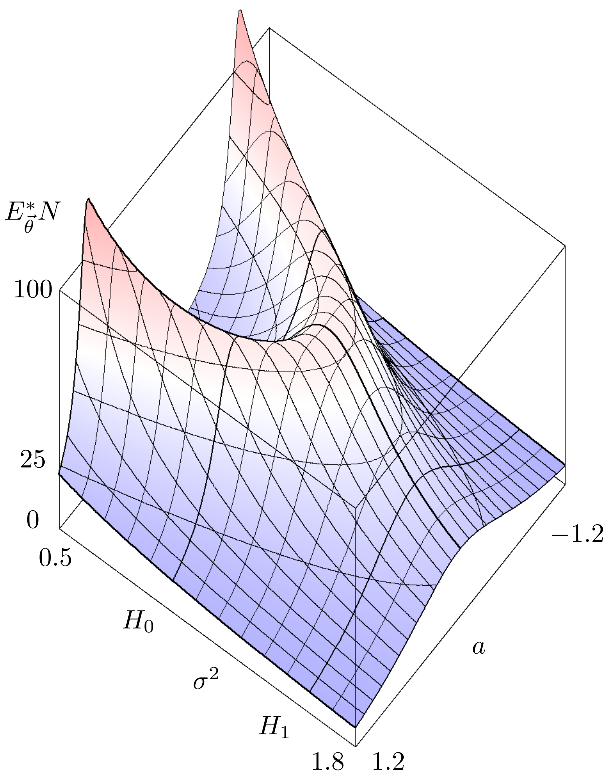


Fig. 10. ASN-function $E^*_{a,\sigma^2}N$ for $H_0 : a_0 = 0, \sigma_0^2 = 1, H_1 : a_1 = 0, \sigma_1^2 = 1.6$.

is a density function. In this case we have

$$\int_{-\infty}^{\infty} f_{a',\sigma'^2}(x)\, dx = E_{a,\sigma^2} \mathrm{e}^{hZ_{1,a_0,\sigma_0^2,a_1,\sigma_1^2}} = 1$$

where $\varphi_{a,\sigma^2}(h) = E_{a,\sigma^2} \mathrm{e}^{hZ_{1,a_0,\sigma_0^2,a_1,\sigma_1^2}}$ for $-\infty < h < \infty$ is the moment-generating function of $Z_{1,a_0,\sigma_0^2,a_1,\sigma_1^2}$. It can by shown, that in case of $P_{a,\sigma^2}(Z_{1,a_0,\sigma_0^2,a_1,\sigma_1^2} = 0) < 1$ this function is a convex function in $h$, $-\infty < h < \infty$ with $\lim_{h\to\pm\infty} \varphi_{a,\sigma^2}(h) = \infty$. Because of $\varphi'_{a,\sigma^2}(0) = E_{a,\sigma^2} Z_{1,a_0,\sigma_0^2,a_1,\sigma_1^2}$ and $\varphi_{a,\sigma^2}(0) = 1$ equation $\varphi_{a,\sigma^2}(h) = 1$ has a unique, non-zero solution $h$. This is again the parameter value $h$ needed in context of our principle of conjugated parameter pairs.

Some examples: Without loss of generality it can be assumed that $a_0 = 0$ and $\sigma_0^2 = 1$. Otherwise, this can be reached by transformation of $X_1, X_2, ...$ according $X_i' = \frac{X_i - a_0}{\sigma_0}$, $i = 1, 2, ....$ The Figures 1-10 demonstrate the behavior of OC- and ASN-function of SPRTs for

$$H_0 : a = 0, \sigma^2 = 1 \quad \text{and} \quad H_1 : a = a_1, \sigma^2 = \sigma_1^2$$

depending on different alternative hypotheses to given error probabilities $\alpha = \beta = 0.05$ of an error of first or second kind and stopping boundaries $B = B^*$ and $A = A^*$ according to (6). Interesting is the behavior of OC-function in case of $a_0 = a_1$, see Fig. 7, which corresponds the one-dimensional test for the variance with known mean. For small or large values of parameter $a$ this test tends for $\sigma^2 < \sigma_0^2$ again to the acceptance of hypothesis $H_1 : \sigma^2 = \sigma_0^2$. Moreover, the examples considered here show how in certain cases tests for composite hypotheses $H_0 : (a, \sigma^2) \in G_0$ against $H_1 : (a, \sigma^2) \in G_1$, $G_0 \cap G_1 = \emptyset$, can be reduced to tests for proper chosen simple hypotheses.

## REFERENCES


[1] V.I. Denisov, K.-H. Eger, B.Yu. Lemesko, and E.B. Tsoy, *Design of experiments and statistical analysis for grouped observations.* Novosibirsk, NSTU Publishing House, 2004.

[2] K.-H. Eger, "Likelihood ratio tests for grouped observations", Chemnitz University of Technology, Faculty of Mathematics, Preprint 2003-10, 2003.

[3] K.-H. Eger, *Sequential tests.* Teubner, Leipzig, 1985.

[4] K.-H. Eger, E.B. Tsoy, "Sequential probability ratio tests based on grouped observations", Proceedings of the Second International Forum on Strategic Technology - IFOST 2007, pp. 284-287, Ulaanbaatar, Mongolia, 2007.

[5] B.K. Ghosh, *Sequential Tests of Statistical Hypotheses.* Addison-Wesley, Reading, Mass, 1970.

[6] B.K. Ghosh, P.K. Sen (editors), *Handbook of Sequential Analysis.* Marcell Dekker, Inc., New York, 1991.

[7] A. Wald, *Sequential Analysis.* Wiley, New York, 1947.


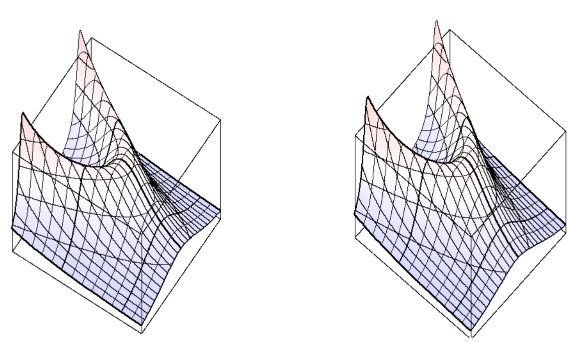

Fig. 11. Three dimensional version of Fig. 10.